Borel classes of closed subgroups of S_∞

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Non-Archimedean groups

All topological groups G in this talk will be Polish. Note that each open subgroup is closed, and has countable index in G.

G is non-Archimedean if it has a countable basis of neighbourhoods of the identity consisting of open subgroups. Such groups have a basis of clopen sets.

They are, up to homeomorphism, the closed subgroups of the topological group S_∞ of permutations of $\mathbb{N}$ with the usual topology of pointwise convergence.

Equivalently, they are the automorphism groups of structures with domain $\mathbb{N}$. (All structures in this talk have domain $\mathbb{N}$.)

Borel classes of closed subgroups of S_∞

The closed subgroups G of S_∞ form a "standard Borel space":

- If σ is a string let $[\sigma] = \{\pi \in S_\infty \colon \sigma \prec \pi\}$.
- The σ-algebra of Borel sets is generated by the sets

$$\{G \colon G \cap [\sigma] \neq \emptyset\}.$$

Programme (Kechris, N. and Tent, 2018; Logic Blog 2020)

(a) For natural classes of closed subgroups of S_∞, determine whether they are Borel.

(b) If a class $\mathcal{C}$ is Borel, study the relative complexity of the topological isomorphism relation, using Borel reducibility $\leq_B$.

I. Borel duality for

locally Roelcke precompact groups

and oligomorphic groups

Largest $\mathcal{C}$: locally Roelcke precompact groups

By G we always denote a closed subgroup of S_∞.
Note that G is compact iff each open subgroup has only finitely many (left) cosets.

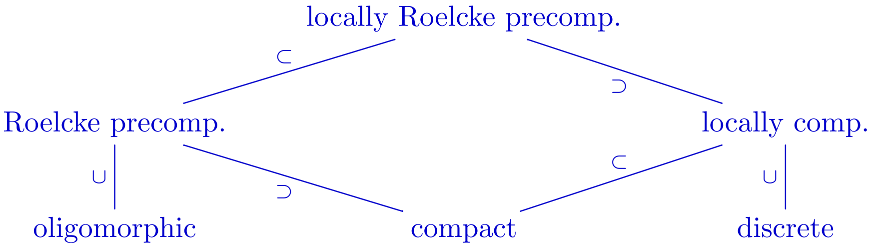
Definition

- G is Roelcke precompact (R.p.) if each open subgroup U has only finitely many double cosets.
- G is locally Roelcke precompact if it has a Roelcke precompact open subgroup.

Let T_∞ be the undirected tree with each vertex of infinite degree.

- $\mathrm{Aut}(T_\infty)$ is locally R.p. (Zielinski), and not locally compact.
- The stabiliser of a vertex is Roelcke precompact.

Some Borel classes $\mathcal{C}$, and inclusion relations



Kechris, N. and Tent, 2018 results :

- Isomorphism relation on each class in the diagram is $\leq_B$ graph isomorphism.
- Fix prime $p \neq 2$. Let $\mathcal{C}$ be the nilpotent-2 profinite groups of exponent p^2.
  topological isomorphism on $\mathcal{C}$ $\geq_B$ graph isomorphism.

Complexity of $\cong$ for oligomorphic groups

$G \leq_c S_\infty$ is called oligomorphic if for each k, the action of G on $\mathbb{N}^k$ only has finitely many orbits. ($G \cong \mathrm{Aut}(M)$ for some $\aleph_0$-categorical M.)

An equivalence relation on a Borel space is essentially countable if it is $\leq_B$ a Borel equivalence relation with all classes countable.

Theorem (N, Schlicht and Tent, J. Math Logic 2021)

The topological isomorphism relation between oligomorphic groups is essentially countable.

Lemma (towards the theorem)

The bi-interpretability relation between $\aleph_0$-categorical theories is essentially countable.

Question

Is this equivalence relation smooth?

$\mathcal{M}$ turns locally R.p. group into coarse group

Given a locally R.p. G, let $\mathcal{M}(G)$ be its coarse group (KNS 2018):

- The domain consists of (numbers encoding) the Roelcke precompact open cosets in G.
- Ternary relation "$AB \subseteq C$" on the domain approximates the group multiplication.

R.p. open cosets approximate elements of G, so this ternary relation approximates the binary group operation.

Each R.p. open subgroup of G is a finite union of double cosets of a basic open subgroup. So there are only countably many such subgroups.

Using descriptive set theory, we can view the operator $\mathcal{M}$ as a Borel function from locally R.p. groups to structures with domain $\mathbb{N}$.

$\mathcal{G}$ turns from coarse groups into locally R.p. group

- Recall that $\mathcal{M}(G)$ is the coarse group of a locally R.p. G.
- Among the structures on $\mathbb{N}$ with a ternary relation, let $\mathbf{CG}$ be the closure under isomorphism of the range of $\mathcal{M}$.
- Write the ternary relation suggestively as "$AB \sqsubseteq C$".

Definition

Given a structure $M \in \mathbf{CG}$, let $\mathcal{G}(M)$ be the closed subgroup of S_∞ consisting of the permutations p such that

$$AB \sqsubseteq C \Longleftrightarrow p(A)B \sqsubseteq p(C) \text{ for each } A, B, C \in M.$$

Recall that we have defined maps $\mathbf{LRP} \underset{\mathcal{G}}{\overset{\mathcal{M}}{\rightleftarrows}} \mathbf{CG}$.

Note that $\mathcal{M}$ and $\mathcal{G}$ forward-preserve (topological) isomorphism.

Theorem (Borel duality for l.R.p. groups)

- $\mathbf{CG}$ is a Borel class. $\mathcal{M}$ and $\mathcal{G}$ are Borel maps.
- $\mathcal{M}$ and $\mathcal{G}$ are inverses up to isomorphism:
  For each $G \in \mathbf{LRP}$ and each $M \in \mathbf{CG}$,
  $$\mathcal{G}(\mathcal{M}(G)) \cong_{top} G \text{ and } \mathcal{M}(\mathcal{G}(M)) \cong M.$$

(Functors showing equivalence of categories.)

As a consequence, for $G_0, G_1 \in \mathbf{LRP}$ and $M_0, M_1 \in \mathbf{CG}$, we have

$$G_0 \cong_{top} G_1 \iff \mathcal{M}(G_0) \cong \mathcal{M}(G_1)$$
$$M_0 \cong M_1 \iff \mathcal{G}(M_0) \cong_{top} \mathcal{G}(M_1)$$

The case of oligomorphic groups G

Note that an oligomorphic G (and hence each open subgroup) is Roelcke precompact.

Theorem (N., Schlicht, Tent, '21)

Among structures on $\mathbb{N}$ with a ternary relation symbol, let $\mathcal{D}$ be the closure of $\{\mathcal{M}(G)\colon G \text{ is oligomorphic}\}$ under isomorphism.

(a) The class $\mathcal{D}$ is Borel.

(b) $\cong_{top}$ on the oligomorphic groups is Borel equivalent with the isomorphism relation on $\mathcal{D}$.

Theorem (N., Schlicht, Tent, '21)

Among structures on $\mathbb{N}$ with a ternary relation symbol, let $\mathcal{D}$ be the closure under isomorphism of $\{\mathcal{M}(G) \colon G \text{ is oligomorphic}\}$.

(a) The class $\mathcal{D}$ is Borel.

(b) $\cong_{top}$ on the oligomorphic groups is Borel equivalent with the isomorphism relation on $\mathcal{D}$.

(a) is proved by a suitable axiomatisation of the class $\mathcal{D}$ using an infinitary language. Start by defining U to be a subgroup if $UU \sqsubseteq U$; A is a left coset of U if U is greatest such that $AU \sqsubseteq A$; etc.

(b) is obtained via Borel duality for oligomorphic groups: introduce a modification $\widehat{\mathcal{G}}$ of the "reverse" Borel operator $\mathcal{G}$ so that $\widehat{\mathcal{G}}(M)$ is oligomorphic for $M \in \mathcal{D}$. This is defined like $\mathcal{G}(M)$ but only for permutations of the left cosets of W, where W is a "faithful" subgroup in M. Under suitable axioms, W can be obtained in a Borel way.

Theorem (N, Schlicht, Tent, JML 2021)

The relation $\mathrm{BI}_{\omega-\mathrm{cat}}$ of Bi-interpretability on ω-categorical structures is essentially countable.

To show this we combine Borel duality for oligomorphic groups with the following:

Theorem (Hjorth and Kechris, APAL 1997, Th. 3.8)

- Let $\mathcal{D}$ be a Borel class of structures with domain $\mathbb{N}$.
- Suppose that $\cong_{\mathcal{D}}$ is potentially F_σ; that is, $\cong_{\mathcal{D}} \leq_B L$ for some F_σ equivalence relation L on some Polish space Y.

Then $\cong_{\mathcal{D}}$ is essentially countable.

In our setting $\mathcal{D}$ is the class of coarse groups above; we've shown that $\cong_{\mathcal{D}}$ is Borel equivalent to $\mathrm{BI}_{\omega-\mathrm{cat}}$.
We verify the hypothesis on $\mathcal{D}$ by showing that $\mathrm{BI}_{\omega-\mathrm{cat}}$ is F_σ.

Complexity of bi-interpret. for $\aleph_0$-cat. theories

Given $\aleph_0$-cat. theories S, T, the potential bi-interpretations can be seen as points in a totally disconnected, locally compact space that is effectively given in $S \oplus T$ as an oracle. One presents the space as the set of infinite paths $[R]$ on a rooted tree R.

The root is infinitely branching; each node at level 1 tells us of the two kinds of imaginaries used (of dimensions r and s), and the two definable isomorphisms.

Above a node at level 1, we have a finitely branching tree. The $k+1$-th level tells how a k-type of S is given as a $k \cdot r$ type of T, and how a k-type of T is given as a $k \cdot s$ type of S.

The tree and its width at each level is computable in $(S \oplus T)'$; so nonemptiness of $[R]$ is Σ^0_2 in $(S \oplus T)'$ by Koenig's lemma.

Extension to quasi-oligomorphic groups

A closed subgroup G of S_∞ is called quasi-oligomorphic if it is isomorphic to an oligomorphic group.
For instance, if G is oligomorphic, then $G/Z(G)$ is quasi-oligomorphic: this is shown by the action of G on $\mathbb{N} \times \mathbb{N}/\sim_{Z(G)}$. (Note that $Z(G)$ is finite.)

Corollary (NST, 2021)

The class of quasi-oligomorphic groups is Borel.
Its isomorphism relation is essentially countable.

Proof idea:

- $\mathcal{M}(G)$ is defined for any Roelcke precompact group.
- $\widehat{\mathcal{G}}(\mathcal{M}(G))$ is oligomorphic via its natural embedding into S_∞.

Paolini: smoothness for w.e.i. theories

- An ω-categorical theory T has weak elimination of imaginaries if for each $M \models T$ and each $e \in M^{eq}$, there is a real tuple $\overline{c}$ such that $e \in \mathrm{dcl}^{eq}(\overline{c})$ and $\overline{c} \in \mathrm{acl}(e)$.
- For $G = \mathrm{Aut}(M)$ this means that each open subgroup U has a subgroup V of finite index that is the pointwise stabiliser of a finite set.
- Borel property of theories.

Theorem (Paolini, 2023)

The bi-interpretability relation is smooth for the Borel class of $\aleph_0$-categorical theories that have weak elimination of imaginaries.

Work in progress with P. would also show this for theories without algebraicity, and perhaps essentially finite theories.

II. Computable duality for

totally disconnected,

locally compact (tdlc) groups

Melnikov and N., 2022, 2204.09878, 45 pages

Basic fact on tdlc groups

Van Dantzig's theorem (1936): Each tdlc group G has a basis of neighbourhoods of 1 consisting of compact open subgroups.

- In particular, every countably based tdlc group is non-Archimedean.
- So the tdlc groups form a proper subclass of the non-Archimedean locally R.p. groups we studied earlier.

Van Dantzig's theorem follows from two facts:

- For each totally disconnected, locally compact space, the clopen sets form a basis.
- For each Hausdorff group, each compact open neighbourhood of 1 contains a compact open subgroup.

Some examples of tdlc groups G

- All profinite groups and all discrete groups.
- $(\mathbb{Q}_p, +)$, the additive group of p-adic numbers for a prime p.
- The semidirect product $\mathbb{Z} \ltimes \mathbb{Q}_p$ where $g \in \mathbb{Z}$ acts as $x \mapsto xp$ on $\mathbb{Q}_p$.
- The groups $\mathrm{SL}_n(\mathbb{Q}_p)$ for $n \geq 2$.
- $\mathrm{Aut}(T_d)$, the automorphisms of an undirected tree with each vertex of degree d. Stabilizer of a vertex is a compact open subgroup.

Previous work for the subclass of profinite groups

Definition

A profinite group G is called computable if

$$G = \varprojlim_i (A_i, \psi_i)$$

for a computable diagram $(A_i, \psi_i)_{i \in \mathbb{N}}$ of finite groups and epimorphisms $\psi_i \colon A_i \to A_{i-1}$ $(i > 0)$.

- This is due to Smith, and la Roche independently, in papers dating both from 1981.
- Smith showed that removing the "epi" makes a strictly weaker notion.

What's a groupoid? (Old notion)

Intuitively, the notion of a groupoid generalizes the notion of a group by allowing that the binary operation is partial.

- A groupoid is given by a domain $\mathcal{W}$ on which a unary operation $(.)^{-1}$ and a partial binary operation, denoted by "$\cdot$", are defined.
- Category view: a groupoid is a small category in which each morphism has an inverse.
- $A\colon U \to V$ means that U, V are idempotent ($U \cdot U = U$), and $A = UA = AV$.

The meet groupoid of a tdlc group G

- $\mathcal{W}(G)$ is an algebraic structure on the countable set of compact open cosets in G, together with $\emptyset$.
- This structure is a partially ordered groupoid. The partial order is set inclusion. We can multiply a left coset A of some subgroup U with a right coset B of the same U. This is a coset because, if $A = aU$ and $B = Ub$ some $a, b \in G$, then
$$AB = aUb = U^{a^{-1}}ab = abU^b.$$
- The intersection of two compact open cosets is either empty or is such a coset itself. So $\mathcal{W}(G)$ is a meet semilattice.

One can define in a first-order way the coarse group from the meet groupoid, and conversely. However, the notions are not computationally equivalent because the definitions need a lot of quantifiers (which amount to unbounded search over the structure).

Abstract definition of a groupoid

Definition

A meet groupoid is a groupoid $(\mathcal{W}, \cdot, (.)^{-1})$ that is also a meet semilattice $(\mathcal{W}, \cap, \emptyset)$ of which $\emptyset$ is the least element.

Writing $A \subseteq B \iff A \cap B = A$, it satisfies the conditions

- $\emptyset^{-1} = \emptyset = \emptyset \cdot \emptyset$, and $\emptyset \cdot A$ and $A \cdot \emptyset$ are undefined for each $A \neq \emptyset$,
- if U, V are idempotents such that $U, V \neq \emptyset$, then $U \cap V \neq \emptyset$,
- $A \subseteq B \iff A^{-1} \subseteq B^{-1}$, and
- if $A_i \cdot B_i$ are defined $(i = 0, 1)$ and $A_0 \cap A_1 \neq \emptyset \neq B_0 \cap B_1$, then

$$(A_0 \cap A_1) \cdot (B_0 \cap B_1) = A_0 \cdot B_0 \cap A_1 \cdot B_1.$$

Automorphism group of G and Chabauty space

Let $\mathcal{W} = \mathcal{W}(G)$ be the meet groupoid of a tdlc group G.

Proposition (Melnikov and N. '22; Logic Blog '23)

- $\mathrm{Aut}(G)$ with the usual Braconnier topology is canonically homeomorphic to $\mathrm{Aut}(\mathcal{W})$:

  Send $\phi \in \mathrm{Aut}(G)$ to its action on $\mathcal{W}$.

- The Chabauty space $\mathcal{S}(G)$ of closed subgroups of G can be canonically represented by a closed subset of $2^{\mathcal{W}}$, consisting of certain ideals of $\mathcal{W}$.

Computably tdlc groups via meet groupoids

A meet groupoid $\mathcal{W}$ is called Haar computable if

(a) its domain is a computable subset D of $\mathbb{N}$;

(b) the groupoid and meet operations are computable; in particular, the relation $\{\langle x, y\rangle \colon x, y \in D \wedge x \cdot y \text{ is defined}\}$ is computable;

(c) the function sending a pair of idempotents $U, V \in \mathcal{W}$ to the number of (left, say) cosets of $U \cap V$ in U has range $\mathbb{N}$ and is computable.

Definition (Computably tdlc groups via meet groupoids)

Let G be a tdlc group. We say that G is computably tdlc if $\mathcal{W}(G)$ has a Haar computable copy $\mathcal{W}$.

We show that this is equivalent to other reasonable definitions.

Algorithmic properties of the scale function

For a compact open subgroup V of G and an element $g \in G$ let

$$m(g, V) = |V^g : V \cap V^g|.$$

Recall the scale function $[T] \to \mathbb{N}$ is

$$s(g) = \min\{m(g, V) \colon V \text{ is a compact open subgroup}\}.$$

E.g., in $\mathbb{Z} \ltimes \mathbb{Q}_p$, where generator $g \in \mathbb{Z}$ acts as $x \mapsto xp$, we have $s(g) = 1$, $s(g^{-1}) = p$.

Fact

The scale function is computably approximable from above.

Example

For $d \geq 3$, the scale function on $\mathrm{Aut}(T_d)$ in the canonical computable presentation is computable.

A noncomputable scale

For the given examples the scale is computable. However:

Theorem (Melnikov, N., Willis, 2022)

There is a computable presentation of a tdlc group G based on a tree T such that the scale function $s\colon [T] \to \mathbb{N}$ is not computable.

In fact, there is a uniformly computable sequence $(g_n)_{n\in\mathbb{N}}$ in G such that $s(g_n) = 2$ if $n \notin \mathcal{K}$ (the halting problem), and 1 otherwise.

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